Multiple solutions in extracting physics information from experimental data

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Abstract Multiple solutions exist in many experimental situations when several interfering amplitudes are summed to fit experimentally measured distributions, such as cross sections, mass spectra, and/or the angular distributions. We show a few examples where multiple solutions are found, but only one solution is reported in the publications. Since there is no standard rule for choosing one among the solutions as the physics one, we propose a simple rule that agrees with what has been adopted in previous literatures: the solution corresponding to the minimal magnitudes of the amplitudes must be the physical solution. We suggest test this rule in the future analyses.

Key words amplitude, multi-solution, mixing, physics solution

1 Introduction

In quantum mechanics, a physics observable is proportional to the squared modulus of the amplitude. In case of more than one amplitude contributing to a process, they are summed to obtain the total amplitude, and thus one generally has contributions from interference terms to the physics observable. It is simple to predict a physics observable when the amplitudes and the relative phases between them are known, in which case there are no ambiguities.

However, in many circumstances, the experimental quantities are measured, and from these one extracts information on the amplitudes. As there is a square operation on the summed amplitudes in calculating the observable, one expects that mathematically, there can be multiple solutions. In these cases one faces a common problem: which solution is the physics one.

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In this paper, we present a few examples where two-solutions are reported in fitting with the coherent sum of two amplitudes; and revisit a few cases where only one solution is reported and where we find other distinct solutions from fits with two amplitudes; we also discuss more complicated situations where more than two amplitudes are summed to fit the data. Finally, we propose a conjecture on how to choose the physics solution among multiple solutions.

2 Examples with two-solutions reported

A few recent examples where multiple solutions are reported are studies of the so-called Y states that are produced via initial state radiation by the Belle experiment. Figure 1 shows the invariant mass distributions of $\pi^+\pi^- J/\psi$ and $\pi^+\pi^-\psi(2S)$ measured in Belle [1, 2], together with a fit with two coherent resonant terms and an incoherent background term. Table 1 shows the fit results, including the $Y(4008)$ and $Y(4260)$ Breit-Wigner (BW) terms from the $\pi^+\pi^- J/\psi$ mode, and the $Y(4360)$ and $Y(4660)$ BW terms from the $\pi^+\pi^-\psi(2S)$ mode. It should be noted that in both channels there are two solutions with exactly the same goodness-of-the-fit ($\chi^2/ndf = 81/78$ for $\pi^+\pi^- J/\psi$ mode [1] and $\chi^2/ndf = 4.7/3$ for $\pi^+\pi^-\psi(2S)$ mode [2]), with exactly the same masses and widths for the resonances but with very different couplings to e^+e^- pair ($\Gamma_{e^+e^-}$). Instead of choosing one of the two solutions, Belle reported both in their publications; however, the Particle Data Group (PDG) frequently uses only one of the solutions in averaging with results from other experiments [3, 4]. Such a treatment of data is rather capricious, since the solutions picked up randomly from different experiments may have distinctive features and, thus, should not be averaged together.

Another example is the study of the decay dynamics of $\eta' \to \gamma\pi^+\pi^-$ mode by BES [5]. When the $\pi^+\pi^-$ invariant mass distribution is fitted with coherent sum of the ρ resonance and a contact term, it is found that there are two solutions of equal goodness-of-the-fit. One solution corresponds to constructive interference between the two amplitudes while the other to destructive interference. However, in previous analyses only one solution [6] was reported, the one that corresponds to the solution with constructive interference in the BES analysis [5].

3 Examples with one-solution reported

Since the examples described in the previous section are basically fits to an s-dependent distribution with two coherent amplitudes, one would suspect that there would be, in general, two solutions in all such circumstances. There are many of these types of fits in the literature, especially in low energy e^+e^- annihilation experiments, where the cross

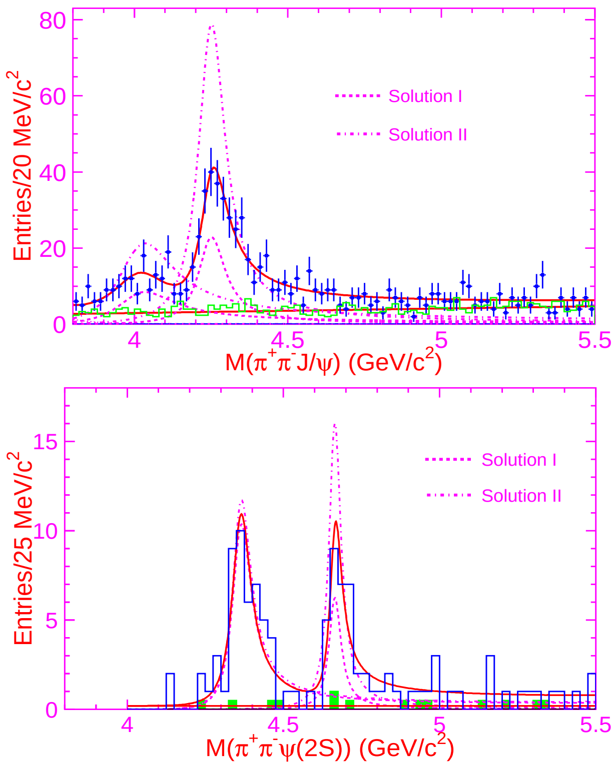

Figure 1: The $\pi^+\pi^- J/\psi$ (upper) and $\pi^+\pi^-\psi(2S)$ (lower) invariant mass distributions and the best fit with two coherent resonances together with a background term. The data are from Belle [1, 2].

section of $e^+e^- \to$ hadrons is parameterized as the coherent sum of the amplitudes of the vector mesons. Below, we show two typical cases where only one solution is reported, and we have redone the fit to obtain the other solution from the data.

3.1 Branching fraction of $\phi \to \omega\pi^0$

The most precise data on $e^+e^- \to \omega\pi^0$ near the ϕ resonance were reported by KLOE [7] for both the $\omega \to \pi^+\pi^-\pi^0$ and $\omega \to \gamma\pi^0$ decay modes. In the KLOE paper [7], the cross section as a function of the center-of-mass energy, $\sqrt{s}$, is parameterized as

$$\sigma(\sqrt{s}) = \sigma_{nr}(\sqrt{s}) \cdot \left| 1 - Z \frac{M_\phi \Gamma_\phi}{D_\phi(\sqrt{s})} \right|^2 ,$$

where $\sigma_{nr}(\sqrt{s}) = \sigma_0 + \sigma'(\sqrt{s} - M_\phi)$ is the bare cross section for the non-resonant process, parameterized as a linear function of $\sqrt{s}$; Z is the interference parameter, while M_ϕ, Γ_ϕ, and $D_\phi = M_\phi^2 - s - iM_\phi\Gamma_\phi$ are the mass, the width, and the inverse propagator of the ϕ meson, respectively.

We take the KLOE data for $\omega \to \pi^+\pi^-\pi^0$ from Table I of Ref. [7] and fit with the

Table 1: Fit results of the $\pi^+\pi^- J/\psi$ and $\pi^+\pi^-\psi(2S)$ invariant mass spectra [1, 2]. The first errors are statistical and the second systematic. M, $\Gamma_{\rm tot}$, and $\mathcal{B}\cdot\Gamma_{e^+e^-}$ are the mass (in MeV), total width (in MeV), product of the branching fraction to hadronic mode and the e^+e^- partial width (in eV), respectively. ϕ is the relative phase between the two resonances (in degrees).

Parameters	Solution I	Solution II
$M(Y(4008))$	$4008 \pm 40^{+114}_{-28}$	
$\Gamma_{\rm tot}(Y(4008))$	$226 \pm 44 \pm 87$	
$\mathcal{B}\cdot\Gamma_{e^+e^-}(Y(4008))$	$5.0 \pm 1.4^{+6.1}_{-0.9}$	$12.4 \pm 2.4^{+14.8}_{-1.1}$
$M(Y(4260))$	$4247 \pm 12^{+17}_{-32}$	
$\Gamma_{\rm tot}(Y(4260))$	$108 \pm 19 \pm 10$	
$\mathcal{B}\cdot\Gamma_{e^+e^-}(Y(4260))$	$6.0 \pm 1.2^{+4.7}_{-0.5}$	$20.6 \pm 2.3^{+9.1}_{-1.7}$
ϕ	$12 \pm 29^{+7}_{-98}$	$-111 \pm 7^{+28}_{-31}$
$M(Y(4360))$	$4361 \pm 9 \pm 9$	
$\Gamma_{\rm tot}(Y(4360))$	$74 \pm 15 \pm 10$	
$\mathcal{B}\cdot\Gamma_{e^+e^-}(Y(4360))$	$10.4 \pm 1.7 \pm 1.5$	$11.8 \pm 1.8 \pm 1.4$
$M(Y(4660))$	$4664 \pm 11 \pm 5$	
$\Gamma_{\rm tot}(Y(4660))$	$48 \pm 15 \pm 3$	
$\mathcal{B}\cdot\Gamma_{e^+e^-}(Y(4660))$	$3.0 \pm 0.9 \pm 0.3$	$7.6 \pm 1.8 \pm 0.8$
ϕ	$39 \pm 30 \pm 22$	$-79 \pm 17 \pm 20$

parametrization given above. In our fit, the Born-order cross section is calculated as $\sigma_{\rm vis}/\delta_{\rm rad}$, where $\sigma_{\rm vis}$ is the visible cross section and $\delta_{\rm rad}$ is the radiative correction factor; only statistical errors are considered in the χ^2 construction. Table 2 shows the results from our fit, where two solutions are found with the same fit quality ($\chi^2/ndf = 4.3/13$). We can see that the parameters from Solution I are very close to those listed in Table II of Ref. [7], the slight difference is due to the simplified procedure for the Born-order cross section calculation in our fit. It can be seen that Solution II has a much larger resonant amplitude, and the resulting branching fraction of the isospin violating process $\phi \to \omega\pi^0$ is at per mille level, about two orders of magnitude higher than Solution I, namely, the solution reported in the original work [7]. Figure 2 shows the fit results and the contribution of each component in the fit. Similarly, we made a fit to the cross sections measured with $\omega \to \gamma\pi^0$, there are also two solutions obtained from the fit and these results are in good agreement with those from $\omega \to \pi^+\pi^-\pi^0$.

Table 2: Results from fits to the $e^+e^- \to \omega\pi^0$ cross sections measured with $\omega \to \pi^+\pi^-\pi^0$.

Parameter	Solution I	Solution II
σ_0 [nb]	7.88 ± 0.04	7.88 ± 0.08
$\Re(Z)$	0.106 ± 0.004	0.106 ± 0.006
$\Im(Z)$	-0.103 ± 0.003	-1.90 ± 0.006
σ' [nb/MeV]	0.064 ± 0.002	0.064 ± 0.006
$\mathcal{B}(\phi \to \omega\pi^0)$	4.61×10^{-5}	7.62×10^{-3}

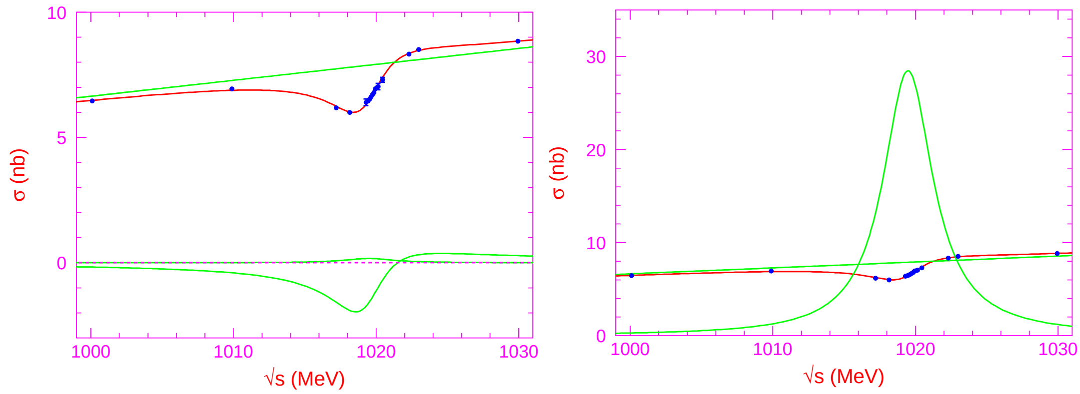

Figure 2: Fit to the $e^+e^- \to \omega\pi^0$ cross sections as a function of center-of-mass energy, $\sqrt{s}$. Data points are shown with dots with error bars. Left plot is for solution I: at $\sqrt{s} = 1.019$ GeV, from top to bottom, the solid curves are continuum, best fit, resonance, and interference terms. Right plot is for Solution II: at $\sqrt{s} = 1.019$ GeV, from top to bottom, the solid curves are resonance, continuum, and best fit, the interference term is not shown.

3.2 F_π and the ρ-ω mixing

Even before the discovery of the τ lepton, Tsai calculated the branching fractions of such a heavy lepton decaying into vector final states such as $\tau^- \to \pi^-\pi^0\nu_\tau$ by applying the conserved vector current (CVC) hypothesis to $e^+e^- \to \pi^+\pi^-$ data measured in e^+e^- annihilation experiments [8]. This calculation, referred to as $\mathcal{B}^{\mathrm{CVC}}_{\tau^-\to\pi^-\pi^0\nu_\tau}$, has been tested ever since the discovery of the τ lepton. As the precision of both the e^+e^- annihilation and the τ decay experiments improved significantly in the past three decades, the test has reached the precision level of better than 1%, and a discrepancy at the 2σ level between the measured $\tau^- \to \pi^-\pi^0\nu_\tau$ branching fraction $\mathcal{B}_{\tau^-\to\pi^-\pi^0\nu_\tau}$ and $\mathcal{B}^{\mathrm{CVC}}_{\tau^-\to\pi^-\pi^0\nu_\tau}$ is observed [9]. Many theoretical efforts have been made to explain this discrepancy, including a better understanding of the isospin violation correction.

One source of isospin breaking is the effect of ρ-ω mixing in the e^+e^- annihilation data that is absent in τ decays. The mixing effect on $\mathcal{B}^{\rm CVC}_{\tau^-\to\pi^-\pi^0\nu_\tau}$ is estimated by fitting the $e^+e^- \to \pi^+\pi^-$ data in the vector meson dominance (VMD) model (including ρ and its excited states and ω) and subtracting the ω contribution by setting its amplitude to zero. This has been carried out in many previous analyses [9, 10, 11, 12], but in all cases only one solution is reported. As a heuristic example, we only use the CMD2 data and follow the fit described in the CMD2 publication [13], namely, with the GS-parametrization [14] of the pion form factor F_π:

$$F_\pi^{\rm GS}(s) \quad = \quad \frac{{\rm BW}^{\rm GS}_\rho(s)\left[1+\delta\frac{s}{m_\omega^2}P_\omega(s)\right]+\beta{\rm BW}^{\rm GS}_{\rho'}(s)}{1+\beta}\,,$$

where

$$\begin{aligned}
{\rm BW}^{\rm GS}_V(s) &= \frac{m_V^2(1+d\cdot\Gamma_V/m_V)}{m_V^2-s+f(s)-im_V\Gamma_V(s)},\\
P_\omega(s) &= \frac{m_\omega^2}{m_\omega^2-s-im_\omega\Gamma_\omega}.
\end{aligned}$$

The definitions of all the quantities can be found in Refs. [11, 12, 14]. Here, the complex numbers δ and β, as well as the mass and width of the ρ, are fit parameters. The masses and widths of ω and ρ' are fixed to their PDG values [3, 4], and the phase of β is fixed at 180°.

We find two solutions with equally good quality ($\chi^2/ndf = 21/25$) of fit, as shown in Fig. 3 and Table 3. It is clear that Solution I is in good agreement with the results reported by CMD2 analysis [13], and the resulting ρ-ω mixing corrections to $\mathcal{B}^{\rm CVC}_{\tau^-\to\pi^-\pi^0\nu_\tau}$ and a_μ (muon magnetic anomaly) are in agreement with the results in Refs. [9, 10]. However, the relative ω and ρ strengths are different in the other solution, as are the corrections to $\mathcal{B}^{\rm CVC}_{\tau^-\to\pi^-\pi^0\nu_\tau}$ and a_μ, as shown in Table 3.

As the amplitude of the ω term in Solution II is more than an order of magnitude larger than that in Solution I, we would expect a branching fraction of $\omega\to\pi^+\pi^-$ much larger than that reported in previous analyses: about 70% in Solution II versus 1.5% in Solution I. It should be emphasized that $\mathcal{B}(\omega\to\pi^+\pi^-\pi^0)$ depends on the measurement of $\omega\to\pi^+\pi^-$, $\omega\to\gamma\pi^0$, and so on, since they all are derived from e^+e^- annihilation experiment, and there are no independent measurements of $\mathcal{B}(\omega\to\pi^+\pi^-\pi^0)$ [3, 4]. The well known $\mathcal{B}(\omega\to\pi^+\pi^-\pi^0)\approx 90\%$ is a result of taking Solution I of the fit to F_π. Solution II seems to be in contradiction with expectations that the isospin breaking decay $\omega\to\pi^+\pi^-$ should be much smaller than the isospin conserving decay $\omega\to\pi^+\pi^-\pi^0$. However, it is also the case that we have come to accept a small $\omega\to\pi^+\pi^-$ branching fraction because the existence of the second solution in the fit has never been considered.

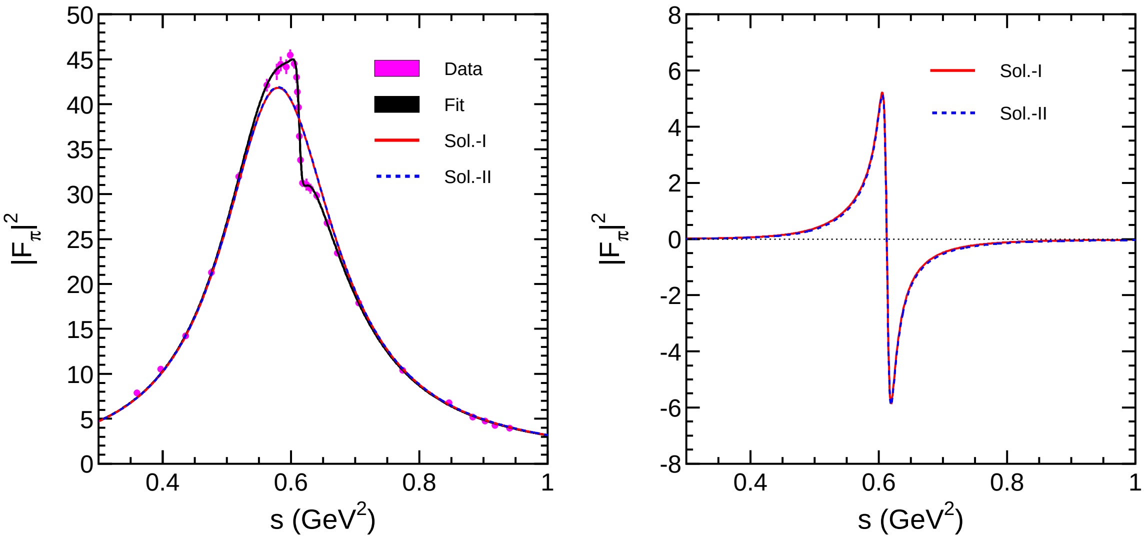

Figure 3: Fit to the $e^+e^- \to \pi^+\pi^-$ form factors below $s = 1$ GeV^2 measured at CMD2 [11, 12]. Left: along the data points (dots with error bars) is the best fit. The $I = 1$ part of the fit is shown for the two solutions. Right: comparison of the ρ-ω interference part in two solutions.

4 Examples with more than two amplitudes

The examples shown above are all described by the sum of two amplitudes with an unknown relative phase in the fit. In the case of more than two amplitudes contributing to the distribution, the number of solutions is 2^{n-1}, where n is the number of free amplitudes.

In the fit to the $e^+e^- \to \phi\pi^+\pi^-$ cross sections [15], incoherent and coherent sums of two amplitudes (the $\phi(1680)$ and $Y(2175)$) are considered, while in estimates of the significance of a third resonance ($X(2400)$), Belle also tested the possibility of the coherent sum of three amplitudes. This procedure is repeated with combined BaBar [16, 17] and Belle data [15] in Ref. [18]. In this latter case, four solutions are found with the same masses and widthes for the resonances, but different coupling constants.

The same phenomenon was also shown in the fit to combined BaBar [19] and Belle [2] data on $e^+e^- \to \pi^+\pi^-\psi(2S)$ [20], where four solutions were found when fitting with the coherent sum of $Y(4360)$, $Y(4660)$, and $Y(4260)$ amplitudes.

5 Partial wave analysis

All the examples discussed so far are one-dimensional problems. In the case of fitting a multi-dimensional distribution, the same multi-solution problem also exists. The partial wave analysis (PWA) and the Dalitz plot analysis techniques used widely in hadron physics potentially have such a problem.

Table 3: Results from fits to the $e^+e^- \to \pi^+\pi^-$ form factors measured at CMD2 [11, 12]. $\Delta\mathcal{B}^{\text{mixing}}$ and $\Delta a_\mu^{\text{mixing}}$ are the corrections to $\mathcal{B}^{\text{CVC}}_{\tau^-\to\pi^-\pi^0\nu_\tau}$ and a_μ, respectively, due to ρ-ω mixing.

Parameter	Solution I	Solution II	Davier [9]
m_ρ [MeV]	775.9 ± 0.5		–
Γ_ρ [MeV]	146.0 ± 0.8		–
$\lvert\delta\rvert$ [$\times 10^{-3}$]	1.62 ± 0.06	21.97 ± 0.04	–
ϕ_δ [°]	10.1 ± 1.4	86.56 ± 0.17	–
$\lvert\beta\rvert$	0.086 ± 0.004		–
$\Delta\mathcal{B}^{\text{mixing}}$ [%]	-0.03 ± 0.01	$+0.04 \pm 0.01$	-0.01 ± 0.01
$\Delta a_\mu^{\text{mixing}}$ [10^{-10}]	$+2.5 \pm 0.2$	$+1.6 \pm 0.2$	$+2.80 \pm 0.19$

In a recent BES analysis of the $\psi(2S) \to \pi^+\pi^- J/\psi$ process [21], the $\pi^+\pi^-$ system is fitted with contributions from σ, a contact term, and a possible tensor amplitude f_2. It is curious that the two very similar models shown in Figure 4 of Ref. [21] give very different strengths for the σ term. The aforementioned expositions led us to suspect that two different solutions are found in each of the two models, one with constructive interference and the other with strong destructive interference. Fortunately, according to the study above, we found that all different solutions result in an identical line shape for the resonance (the same mass and width), and, thus, the resonant parameters are unaffected. However, since the fraction of each component can be very different for different solutions, the branching fraction for each component obtained from these analyses can not be compared directly if they are from different solutions.

A typical partial wave analysis has many amplitudes and all of them are added coherently to fit the data distributions. In principle, there are multiple solutions, although we cannot judge how many-fold the ambiguity may be. In some circumstances, when more than one solution is found, a simple average has been taken to be the best estimate of the parameters. Such a treatment is obviously non-physical, since any one of the solutions is a good description of the data but the average is generally not. In some other cases, the multiple solution problem is just overlooked, which makes the comparison between experiments not meaningful, since one compares two different solutions from two experiments which are different by definition. Therefore, some care must be taken in comparing the branching fractions obtained from fitting multi-dimensional distributions with the coherent sum of several amplitudes. Unless they are from the same solution, the comparison is meaningless.

6 Minimal amplitude conjecture

In the preceding sections, we examined a few cases where amplitudes are extracted from experimentally observed distributions, and in all these cases more than one solution was found. We also discussed the similar problem in PWA. All these indicate that this problem actually is common to many analyses. However, this problem is often overlooked and, more often than not, only one solution is reported with no indication of any reason for neglecting the others. Moreover, as we believe the physics must correspond to only one of these solutions, there is no existing rule or solid physics argument on how one should select one solution from all the possibilities.

However, from the existing examples and our observation, we notice that the solution with the smallest modulus of the amplitude always agrees with our expectation, for example the isospin violating processes $\phi \to \omega\pi^0$ and $\omega \to \pi^+\pi^-$ should be small if ω is an isospin zero dominant state. So it may not be very surprising that the physics solution of any system corresponds to the one where the modula of the amplitudes take the minimal values among all the solutions. We call this selection rule "the minimal amplitude conjecture".

Such a rule can be comprehended readily from a simple "economic" principle. One would expect that a solution in which the physics observable is produced via two or more very large amplitudes with strong destructive interference is less economic than the one via small amplitudes but with constructive interference. In the latter case, we always find a maximum constructive interference among amplitudes. With this minimal amplitude conjecture, we can determine the unique physics solution of all the aforementioned examples. This will make theoretical analyses applicable to cases where many solutions are extracted from experimental data.

7 Conclusion

In this paper, we put forth two important problems: multiple solutions in fitting experimental data and the selection of the physics solution among many possible solutions. As a matter of fact, more and more cases with multiple solutions appear in recent data analyses and even more may have been overlooked previously. Here, we strongly suggest that all possible solutions be found out and reported in the future analyses, in order to avoid a babel of arguments and misleading theoretical deductions.

In theoretical analysis, when confronting with multiple solutions, one has to make a choice. We propose a "minimal amplitude conjecture": the solution with minimal modula of the amplitudes to be the physics one. Such a rule ensures a unique solution be singled out from multiple solutions. This conjecture could further be tested with more

experimental data analyses. If more and more facts are found to support it, it could be promoted to a principle.

Note added: We were reminded by the Journal referee of a preprint arXiv:0710.5627 [22], where the author studied mathematically the ambiguity in determining the resonant parameters in fitting to the cross sections with the coherent sum of two or more Breit-Wigner functions, and obtained analytical relations between different solutions for some special forms of the amplitudes.

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